Journal of Statistical Physics, Vol. 79, Nos. 1/2, 1995

Are Damage Spreading Transitions Generically in the Universality Class of Directed Percolation?

Peter Grassberger[1]

Received September 21, 1994

We present numerical evidence for the fact that the damage spreading transition in the Domany–Kinzel automaton found by Martins *et al.* is in the same universality class as directed percolation. We conjecture that also other damage spreading transitions should be in this universality class, unless they coincide with other transitions (as in the Ising model with Glauber dynamics) and provided the probability for a locally damaged state to become healed is not zero.

KEY WORDS: Damage spreading; directed percolation; stochastic cellular automata; critical behavior; kinetic second-order phase transitions.

Among all critical phenomena, directed percolation (DP) is maybe that which has been associated with the widest variety of phenomena.

First there are interpretations where the preferred direction is a spatial direction. This was of course proposed to apply to material and charge transport in disordered media under the influences of external forces. Also, it should model the propagation of epidemics and forst fires under some directional bias, e.g., strong wind.

More interesting are interpretations where the preferred direction is time. Here, the primary interpretation is as an epidemic without immunization, the so-called "contact proces"[1] or the "simple epidemic."[2]

But these are by no means all the possible applications. A very early application (even if it took rather long until it was understood as such[3,4] was to "reggeon field theory," a theory for ultrarelativistic particle collisions popular in the 1970s.[5] Here, the preferred direction is that of

[1] Physics Department, University of Wuppertal, D-42097, Wuppertal, Germany.

0022-4715/95/0400-0013$07.50/0

"rapidity," while the directions transverse to it are provided by the impact parameter plane. This connection is interesting since it was through it that first precise estimates of critical exponents and amplitudes were obtained for DP.[5]

Another realization of the DP transition occurs in simple models of heterogeneous catalysis. The first such model was proposed by Ziff *et al.*[6] (ZGB). The simulations by these and subsequent authors indicated that this model was in a different universality class, and it was only after some controversy that it became generally accepted to be in the DP universality class.[7] Similar models have been invented again and again.[8,9] Repeatedly they have been claimed to be in different universality classes, and repeatedly these claims have been refuted.[10–12]

In refs. 13 and 14 it was proposed that the universality class of DP contains all continuous transitions from a "dead" or "absorbing" state to an "active" one with a single scalar order parameter, provided the dead state is not degenerate [and provided some technical points are fulfilled: short-range interactions both in space and time, nonvanishing probability for any active state to die locally, translational invariance (absence of "frozen" ramdomness), and absence of multicritical points]. It seems fair to say that there is now ample evidence for this proposal. It predicts, e.g., immediately that the ZGB model is in this universality class. A rather subtle question is whether also chaotic systems where the random noise is replaced by deterministic chaos are in the same class.[15,16]

As far as I am aware of, no model with nonfluctuating absorbing state and a multicomponent order parameter has ever been studied in the literature. Notice that this has to be distinguished from models for which some *mean-field approximation* has a multicomponent order parameter. Such models are quite common (e.g., the Bethe–Peierls approximation of the Ising model, or the mean-field approximation of ZGB), and it was just the study of such a model which had led to the conjecture in ref. 14. A supposed generalization[17] of the above conjecture is thus already fully contained in the original conjecture of ref. 14.

A more interesting question is what happens if the dead state is degenerate. Counterexamples with twofold degeneracy were studied in refs. 18–20. They involve conservation laws whith prevent some active states from dying, making it thus immediately clear that any transition—if it occurs at all—has to be in a different universality class.

But the main open problem is whether models can be generically in the DP class if they have an absorbing state with positive entropy. Here we have to distinguish clearly between two different situations.

In the first class of models, the absorbing states are nonergodic with the number of ergodic components growing exponentially with system size.

In the extreme case, all absorbing (micro-)states can be completely frozen. Examples are the dimer reaction model of refs. 21 and 22 and the dimer–trimer model.[23,21] Here, the absorbing states are indeed "dead" in the sense that they are strictly frozen in the configuration which happened to have been reached after the last active sites died out. But simulations showed that there were no long-range correlations in these dead states, and they were characterized by *unique* statistical properties. The original studies of some such models[23,24] placed them into separate universality classes. But as we pointed out already, systematic errors are often underestimated. More recent simulations found agreement with DP in all cases,[25,21,22,26] provided the initial state had in dead regions the same statistical properties as the states left over after activities died out. This is intuitively plausible, as the absorbing state is essentially unique on a coarse scale, and only coarse-grained properties should influence critical behavior.

In contrast to this are models where there is a single active ergodic component which has, however, positive entropy and fluctuating dynamics.[2] An example for this is the "threshold transfer process" of ref. 22. For obvious reasons, we prefer not to call the absorbing state dead in this case. Assume furthermore that the evolution of the absorbing state is mixing and does not lead to long-range correlations within this state (long correlations should be entirely due to patches of "active" states). For such models it seems even more natural than for the above class that any continuous transitions should in the DP class. For the threshold transfer process this was indeed verified[22] (as explained below, a violation of universality for "dynamic" properties seen in ref. 22 should not be considered as a contradiction to the above).

In this note we propose that there is a rather large and well-studied class of transitions which are exactly of the latter type, and which are thus all in the DP class. These are so-called "damaging" transitions. In these models one considers two replicas of a stochastic spin system, and lets them evolve with *identical realizations* of the stochastic noise. The initial conditions can be either completely independent, or one can start with two states which are identical except for a single spin. This single flip is considered as a "damage," and the question is whether this damage will finally heal, so that both replicas converge toward identical states—or whether it will spread. If the two states are uncorrelated initially, the transition is between a situation where their rescaled Hamming distance (= density of damaged sites) stays finite and one where this distance goes to zero.

[2] In the model studied below we encounter a slight generalization where we have indeed two active ergodic components, but only one of them is realized for nearly all initial conditions. This gives the same behavior, provided some caveats are taken into account as discussed below.

More precisely, we propose that such damaging transitions are in the DP class if they do not coincide with another transition (since then there would be long-range correlations in the absorbing state), and if there is no frozen randomness. The former applies to the 2D Ising model with Glauber dynamics, since there the damaging transition coincides with the ordinary critical point[27] (the situation is less clear in three dimensions[28,29]). Frozen randomness is involved in damage in spin glasses[30–32] and in the extensive studies of damage in Kauffman models.[33–35] This should be in the same universality class as DP with frozen randomness,[36] but for the Kauffman models there is a further complication: there damage typically does not heal completely, whence the "dead" state is not absorbing in our sense.[3] We might mention that it was already pointed out that damage in the *annealed* Kaufmann model is in the DP class,[34] but this is much more trivial than our present claim. The annealed model can be mapped *exactly* onto DP, which is not the case in general.

We support our claim with simulations of damage in the Domany–Kinzel cellular automaton (CA).[38] This is a CA with one space and one time dimension, and with two states per site: $s_i = 0, 1$. Dynamics is defined by the following rule involving two real parameters p_1 and p_2 (we make a trivial modification which slightly simplifies the simulation):

(i) If $s_i = 0$ and $s_{i+1} = 0$, then $s'_i = 0$.

(ii) If $s_i \text{ XOR } s_{i+1} = 1$, then $s'_i = 1$ with probability p_1 and $s'_i = 0$ with probability $1 - p_1$.

(iii) If $s_i \text{ AND } s_{i+1} = 1$, then $s'_i = 1$ with probability p_2 and $s'_i = 0$ with probability $1 - p_2$.

For $p_1 < 1/2$ and any $p_2 < 1$, it is obvious that any state will converge toward the dead state ...000.... Actually, this state is an attractor for all values of p_1 below a critical curve $p_1^c(p_2)$. This curve is indicated as curve $\mathscr{C}$ in Fig. 1. To the right of $\mathscr{C}$, one has an active state (the dead state still is stationary, but it no longer attracts all initial states) with $\rho \equiv \langle s_i \rangle > 0$.

The above conjecture suggests that the transition all along $\mathscr{C}$ is in the DP class, except at its upper limit point $(p_1, p_2) = (1/2, 1)$, where the model is a discrete-time variant of the exactly solvable voter model[1] ("compact directed percolation"[39]). This is supported by all numerical evidence[40,41] (except for a renormalization group analysis and Monte Carlo simulations presented in ref. 42; in high-precision Monte Carlo

[3] After submission of this paper, it was pointed out to me that Obukhov and Stauffer (37) had already conjectured that damage spreading in Kauffman models might be in the DP class. But they also pointed out the problem that damage in these models typically does not heal even if it does not spread.

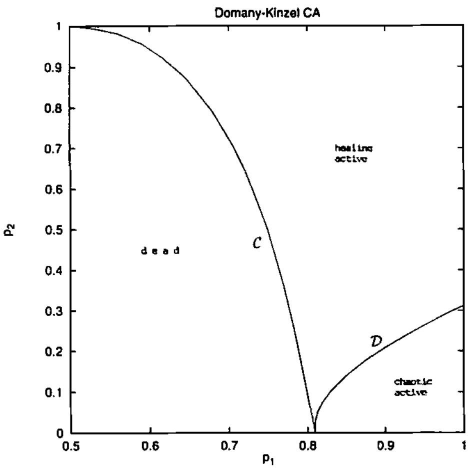

Fig. 1. Part of the phase diagram for the Domany–Kinzel CA. Curve $\mathscr{C}$ separates dead (left) from active (right) phases. Curve $\mathscr{D}$ (which joins $\mathscr{C}$ at its lower end point, but runs otherwise entirely in the active phase) separates a healing phase (left) from a chaotic phase (right) were any damage has nonzero probability not to heal. Here the damage is implemented according to the first variant described in the text. With the second variant, $\mathscr{D}$ would be somewhat further to the left. The transition curves were determined by runs with single active/damaged initial sites, and demanding that the exponent η [see Eq. (4)] has the value of DP. The precision of the curves is everywhere better than the thickness of the lines. Quantitatively, the phase diagram agrees with data from ref. 44, but not with the diagram given in ref. 43. It also deviates significantly from that in ref. 48.

simulations[12] we could not confirm these claims). In particular, bond and site DP correspond to $p_2 = (2 - p_1) p_1$ and $p_1 = p_2$, respectively.

It was found recently in ref. 43 that the active phase can be further subdivided into a phase in which damage does not spread ("healing active phase") and one where it does ("chaotic"). The transition between these two phases is indicated by curve $\mathscr{D}$ in Fig. 1. It corresponds to $p_1 = p_1^d(p_2)$, where $p_1^d > p_1^c$ for all $p_2 > 0$, while $p_1^d(0) = p_1^c(0)$.[44] Indeed, one can consider two different variants of the damage process: in the first one uses different random numbers when applying rules (ii) and (iii) above, and in the second one uses the same. Curve $\mathscr{D}$ is computed with the second variant. The first variant would give a different curve slightly to the left of $\mathscr{D}$.[4]

[4] The very existence of these two variants shows that it is misleading to speak of different phases in the Domany–Kinzel CA, as done in ref. 43. Instead these are different phases for very specific algorithms for simulating pairs of such automata.

As pointed out in ref. 41, one can describe a pair of replicas by an extended phase space with four states per site: (00), (01), (10), and (11). Damage spreading corresponds then to the (directed) percolation of states (10) and (01), while any state with (00) and (11) only is healed. Since $p_1^d > p_1^c$ for all $p_2 > 0$, the healing state has positive entropy at the damage spreading transition, and it does not immediately follow from the conjecture of refs. 13 and 14 that this transition is in the DP universality class.

To check our conjecture that it is in this class nevertheless, we performed extensive simulations at $p_1 = 1$, where both variants coincide. Less extensive runs were make at several other values of p_1, where we studied both variants.

We worked on lattices of length L with periodic boundary conditions. To speed up our simulations, we simulated 64 lattices simultaneously (we worked on machines with 64-bit words) by assigning the kth bit of the ith word in an integer array of length L to the spin $s_i^{(k)}$ in the kth lattice. The dynamics is then easily implemented by standard bit operations.

To measure the degree of damage in simulations which start with independent random initial configurations (thus with half of the sites damaged initially), we count the number of "1" bits in each word. If this number is n_i for the ith word, then the number of pairs of lattices which are damaged at site i is $(64 - n_i) n_i$. The sum of Hamming distances between all $64 \times 63/2 = 2016$ pairs of lattices is thus

$$d = \sum_{i=1}^{L} (64 - n_i) n_i \tag{1}$$

For simulations with initial single-site damage, this is not possible since we cannot build an initial state in which *each* pair is damaged at only one site. Instead, we introduced single-site damage only between successive bits, i.e., we initially damaged the $(2k+1)$th bit $(k = 0, 1, 2, ..., 31)$ in 32 different words, and counted how often the $(2k+1)$th bit differed from the $(2k)$th one.

Results from runs with random and independent initial states on lattices of size $L = 2^{22}$ are presented in Fig. 2. There we show the total damage as function of time for $p_1 = 1$ and several values of p_2. At the critical point we expect an algebraic decay, corresponding to a straight line in Fig. 2. If the transition is in the DP class, this decay is governed by an exponent $\delta = 0.1596 \pm 0.0001$.[45–47] We see indeed a nearly perfect straight line for $p_2 \approx 0.3122$. Together with the data described below, this gives our estimate

$$p_2^d = 0.31215 \pm 0.00004 \qquad \text{for} \quad p_1 = 1 \tag{2}$$

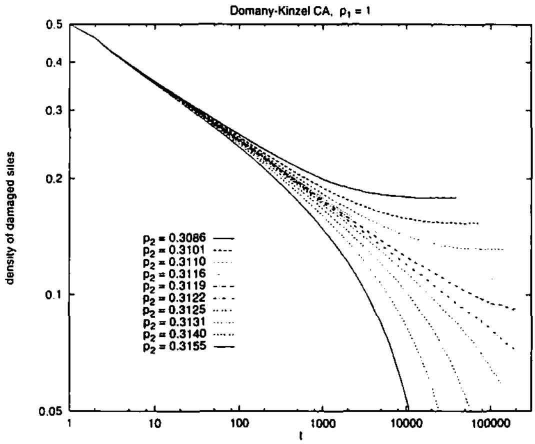

Fig. 2. Log–log plot of the total number of damaged sites in 2016 pairs of lattices, with 2^{22} sites each. For all curves $p_1 = 1$, while p_2 ranges from 0.3086 to 0.3155 (from top to bottom). Initial configurations were random.

and the exponent extracted from it (0.157 ± 0.002) is in good agreement with DP. Similar results (although somewhat less precise) were obtained for both variants of the damage spreading at several values of p_1. For $p_1 = 0.85$, e.g., they gave $p_2^d = 0.1957 \pm 0.0002$ (variant 1), resp. $p_2^d = 0.1400 \pm 0.0002$ (variant 2). In all cases δ was compatible with the DP value.

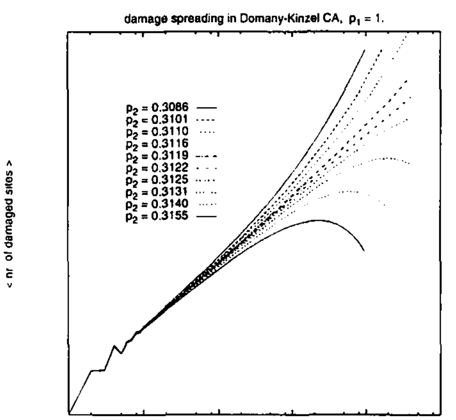

Fig. 3. Log–log plot of the total number of damaged sites in runs where each pair of lattices was initially damaged at a single site. Again $p_1 = 1$. The values of p_2 are the same as in Fig. 2.

It is well known from studies of DP that the exponent β defined by

$$d \sim (p_2^d - p_2)^\beta \tag{3}$$

is not easily measured precisely due to the very long transients close to the critical point (i.e., due to the smallness of δ) and due to finite-size effects. The latter are absent in our simulations due to the very large lattice size. Nevertheless, extrapolating the data from Fig. 2 to $t \to \infty$ gave only a crude estimate $\beta = 0.272 \pm 0.006$, which is, however, in perfect agreement with DP, where $\beta = 0.2766 \pm 0.0003$.[45–47]

In order to measure an independent critical exponent, we made in addition runs with initial single-site damage on much smaller lattices ($L \leqslant 7000$) and for shorter times ($t \leqslant 40{,}000$). Apart from the damaged sites, the initial configurations were randomly chosen active states (they were set to the final configuration of the first lattice in the preceding run by setting the ith word to 0 if $s_i^{(1)} = 0$, and to -1 if $s_i^{(1)} = 1$). From universality with DP we expect that at the critical point

$$d \sim t^\eta, \qquad \eta = 0.314 \pm 0.001 \tag{4}$$

which is nicely fulfilled. Off the critical point we should have

$$\langle d \rangle \propto t^{-\delta} \phi((p_2^d - p_2)\, t^{1/\nu_{\|}}) \qquad \text{(full initial damage)} \tag{5}$$

and

$$\langle d \rangle \propto t^\eta \psi((p_2^d - p_2)\, t^{1/\nu_{\|}}) \qquad \text{(single-site initial damage)} \tag{6}$$

with universal scaling functions $\phi(za)$ and $\psi(z)$ which are regular at $z = 0$, and with $\nu_{\|} = 1.7336 \pm 0.0005$. To see that our data are fully consistent with this, in Fig. 4 we plotted d/t^η against $(p_2^d - p_2)\, t^{1/\nu_{\|}}$ for both types of initial conditions. We see indeed a perfect data collapse as predicted by the above ansatz. We just mention that similar results (again with somewhat smaller statistics and with significantly larger corrections to scaling) were obtained for several other values of p_1, and allowed us to locate curve $\mathscr{D}$ in Fig. 1 with high precision.

In conclusion, we have given numerical evidence that the damage spreading transition in the Domany–Kinzel CA is in the DP universality class, although the undamaged state has positive entropy. We expect this to be true in general, not only for the Domany–Kinzel CA.

Of course, we have to set initial conditions such that we are not confined to atypical states carrying zero measure. In the present case, such atypical behavior would, e.g., result if we would start with one of the configurations being dead (all $s_i = 0$) or nearly dead ($s_i \neq 0$ only in a finite

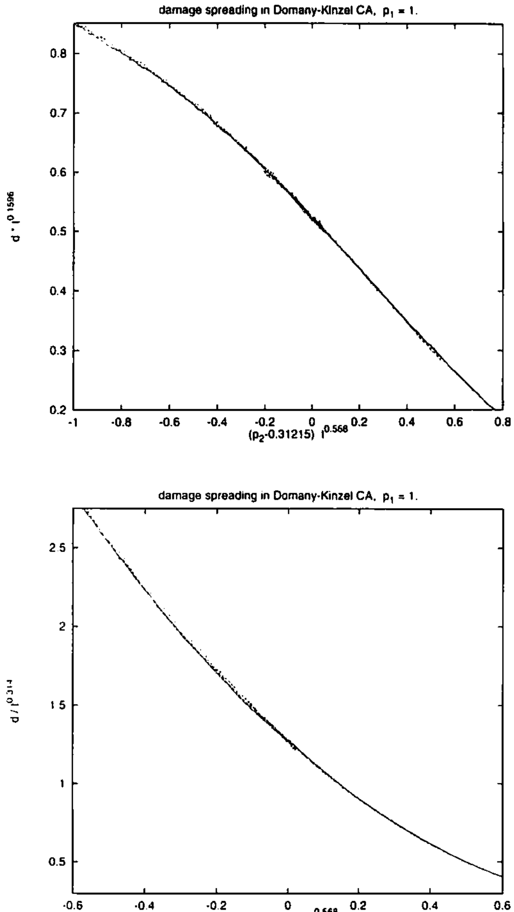

Fig. 4. The same data as in (A) Fig. 2 and (B) Fig. 3, but plotted such that all data should collapse onto a single curves if Eqs. (5) and (6) are correct. Only data for $t > 40$ are plotted in panel (A), and for $t > 10$ in panel (B), in order to reduce finite-time corrections.

region). In the latter case, we would then have a linear increase of d instead of (4). We believe that not taking into account this caveat is the reason why only partial universality with DP was observed for the threshold transfer process in ref. 22. There, "dynamical" simulations were done where the active region was bounded and expanding with time. Outside this region the configurations were not allowed to evolve, but were (artificially) kept in atypical states. It seems trivial that this modification of the model can lead to violations of universality.

Unfortunately, our conjecture does not immediately apply to the case of Kauffman automata,[33] where damage spreading had been studied quite intensively.[34,35] First of all, these models involve frozen randomness and should thus—if at all—be compared to DP in disordered media. Second, healing is not perfect in Kauffman models even in the phase in which damage does not spread. In this phase a finite damage has a nonzero probability to persist forever, and the healed state is not absorbing in our sense. It would be most interesting to study modified Kauffman models where such healing takes place (e.g., stochastic versions—apart from the randomness in the attribution of local rules, Kauffman models are strictly deterministic), and to compare them with DP in disordered media.

We have added one more item to the already long list of possible physical realizations of the DP transition. It is vexing that in spite of this ubiquity in *models*, and in spite of its conceptual simplicity (DP is by far the simplest critical phenomenon to study on a computer and to explain to a high school student), there have not been reported any experiments where the critical behavior of DP was observed even crudely![5] Maybe the present realization can lead to such an observation.

On the more practical side, we have introduced a new and very efficient method for simulating damage spreading which might find applications in similar problems.

ACKNOWLEDGMENTS

I am most indebted to Dr. M. Schreckenberg for interesting discussions and for pointing out the two variants of implementing damage spreading in the Domany–Kinzel CA, and to Dr. T. Lippert for help with running the program on a Thinking Machines CM5. I thank Dr. D. Stauffer for an interesting discussion and for pointing out ref. 37, and to a referee for correcting an important mistake. This work was supported by DFG, SFB 237.

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[5] DP is related to interface pinning in models without overhangs [49,50] in two spatial dimensions. It has been claimed[49] that experiments of 1D fronts in isotropic planar media give agreement with DP. We believe that this is fortuitous, since overhangs are esential in this case (in contrast to moving interfaces!), and lead to pinned interfaces whose scaling at threshold is always described by ordinary 2D percolation.

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